

SPECTRAL INVARIANTS OF THE STOKES PROBLEM

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ABSTRACT. For a given bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, we explicitly calculate the first two coefficients of the asymptotic expansion of the heat trace associated with the Stokes operator as $t \rightarrow 0^+$. These coefficients (i.e., heat invariants) provide precise information for the volume of the domain Ω and the surface area of the boundary $\partial\Omega$ in terms of the spectrum of the Stokes problem. As an application, we show that an n -dimensional ball is uniquely defined by its Stokes spectrum among all Euclidean bounded domains with smooth boundary.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary $\partial\Omega$. We denote by J and V the closures in $[L^2(\Omega)]^n$ and the Sobolev space $[H^1(\Omega)]^n$ respectively of the set of all smooth solenoidal vectors with compact supports in Ω . Let P_J be the orthogonal projection $[L^2(\Omega)]^n \rightarrow J$.

We consider the Stokes operator $S := -\mu P_J \Delta$ with domain $\mathcal{D}(S) = V \cap [H^2(\Omega)]^n$, where Δ is the Laplace operator and μ is the kinematic coefficients of viscosity. The Stokes operator S plays an important role in the investigation of the Navier-Stokes equations since the solution of the Stokes equation provides a good approximation to the solution of nonlinear Navier-Stokes equations (see, [5], [6], [20], [29] and [32]). It is well known (see [1]) that the Stokes operator S generates an analytic semigroup $U(t)$ in space of bounded functions. Furthermore, there exists a matrix-valued function $K(t, x, y)$ (is called the heat kernel or fundamental solution) such that

$$(1.1) \quad (U(t)u_0)(x) = \int_{\Omega} K(t, x, y)u_0(y)dy, \quad u_0 \in V \cap [H^2(\Omega)]^n.$$

Thus, $u(t, x) := (U(t)u_0)(x)$ satisfies the initial-boundary problem for the Stokes equations

$$(1.2) \quad \begin{cases} u_t - \mu \Delta u + \nabla p = 0 & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0 & \text{on } \{0\} \times \Omega, \end{cases}$$

where p is the pressure.

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On the other hand, since the Stokes operator S is an unbounded self-adjoint positive operator in $V \cap [H^2(\Omega)]^n$ with discrete spectrum $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty$, one has

$$(1.3) \quad Su_k = \lambda_k u_k,$$

where $u_k \in V \cap [H^2(\Omega)]^n$ are the corresponding orthogonal eigenvectors. (1.3) can be rewritten as

$$(1.4) \quad \begin{cases} -\mu \Delta u_k + \nabla p_k = \lambda u_k & \text{in } \Omega, \\ \operatorname{div} u_k = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Here the “pressure” term p_k is not known a priori but is determined a posteriori from the solution itself (see, p. 457 of [27]).

The Stokes eigenvalues are physical quantities that can be measured because they just are the frequencies of the vibration of a Stokes (or an incompressible creeping) flow.

An interesting question, which is similar to the famous Kac question for the Dirichlet-Laplacian (see [15], [23] or [33]), is “can one hear the shape of the domain of a Stokes flow by hearing the pitches of its vibration)?”

In [25], Metivier (also see, [2]) proved the Weyl-type asymptotic formula:

$$(1.5) \quad \lambda_k \sim \mu \left(\frac{(2\pi)^n}{(n-1)\omega_n|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \rightarrow \infty,$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n , and $|\Omega|$ denotes the volume of the domain Ω . Kozhevnikov [19] gave the asymptotic formula with sharp reminder estimate:

$$(1.6) \quad N(\tau) = \frac{(n-1)\omega_n|\Omega|}{(2\pi)^n \mu^{\frac{n}{2}}} \tau^{n/2} + O(\tau^{(n-1)/2}) \quad \text{for } \tau \rightarrow +\infty,$$

where $N(\tau)$ is the number of the Stokes eigenvalues less than τ . Each of the formulas (1.5) and (1.6) is equivalent to the first term of asymptotic expansion of the trace of the semigroup $U(t)$:

$$(1.7) \quad \operatorname{Tr}(U(t)) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{(n-1)|\Omega|}{(4\pi\mu t)^{n/2}} \quad \text{as } t \rightarrow 0^+,$$

which implies that one can obtain the volume of the domain Ω if one knows all Stokes eigenvalues.

In this paper, by using a formula of Kozhevnikov [19] for the inverse Stokes operator and by applying a method of pseudodifferential operator, we give the following asymptotic expansion of the trace of the Stokes semigroup:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary $\partial\Omega$, and let $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ be the eigenvalues of the Stokes operator S . Then*

$$(1.8) \quad \sum_{k=1}^{\infty} e^{-\lambda_k t} = \operatorname{Tr}(e^{-tS}) = \frac{(n-1)|\Omega|}{(4\pi\mu t)^{n/2}} - \frac{(n-1)|\partial\Omega|}{(4\pi\mu t)^{(n-1)/2}} + O(t^{-\frac{n}{2}+1}) \quad \text{as } t \rightarrow 0^+.$$

Here $|\partial\Omega|$ denotes the $(n-1)$ -dimensional volume of $\partial\Omega$.

Our result shows that not only the volume $|\Omega|$ but also the surface area $|\partial\Omega|$ can be known if we know all Stokes eigenvalues. Roughly speaking, one can hear the volumes of the domain and the surface area of its boundary $\partial\Omega$ by hearing all the pitches of the vibration of a Stokes flow.

As an application of theorem 1.1, we can prove the following spectral rigidity result:

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that its Stokes spectrum is equal to that of B_r , a ball of radius r . Then $\Omega = B_r$.*

Remark 1.3. Corollary 1.2 shows that a ball is uniquely determined by its Stokes spectrum among all Euclidean bounded domains with smooth boundary.

2. NOTATIONS

Recall that J is the closures in $[L^2(\Omega)]^n$ of the set of all smooth solenoidal vectors with compact supports in Ω . As is known (see [32]), the space J of the vector-valued functions $u = (u_1, \dots, u_n)$ can be rewritten as

$$(2.1) \quad J = \{u \in [L^2(\Omega)]^n \mid \operatorname{div} u = 0, \gamma_N u = (u, N)|_{\partial\Omega} = 0\},$$

where N is the vector of the unit inner normal to $\partial\Omega$. It follows from [32] that the operator $\gamma_N u \equiv (u, N)|_{\partial\Omega}$ continuously maps the Hilbert space $\{u \in [L^2(\Omega)]^n \mid \operatorname{div} u \in [L^2(\Omega)]^n\}$ with inner product $((u, v)) \equiv (u, v) + (\operatorname{div} u, \operatorname{div} v)$ into the space $[H^{-1/2}(\partial\Omega)]^n$. Here (\cdot, \cdot) is the inner product in $[L^2(\Omega)]^n$.

We also introduce the following spaces

$$(2.2) \quad G := \{u \in [L^2(\Omega)]^n \mid u = \operatorname{grad} p, p \in H^1(\Omega), \Delta p = 0\},$$

$$(2.3) \quad F := \{u \in [L^2(\Omega)]^n \mid u = \operatorname{grad} p, p \in H_0^1(\Omega)\}.$$

Then the following orthogonal Weyl-Sobolev decomposition holds (see, Chapter 1 of [32]):

$$(2.4) \quad [L^2(\Omega)]^n = J \oplus F \oplus G,$$

i.e., any vector-valued function $f \in [L^2(\Omega)]^n$ here admits a unique orthogonal decomposition

$$(2.5) \quad f = f_J + f_F + f_G, \quad \text{where } f_J \in J, f_F \in F, f_G \in G.$$

Denote by $\gamma_k u$ the boundary value on $\partial\Omega$ of the derivative

$$(2.6) \quad D_N^k := (-i)^k \partial^k / \partial x_N^k$$

in the direction of the inner normal N to the boundary $\partial\Omega$. We denote by G_1 and G_2 the operators solving the Dirichlet problem for the Poisson and Laplace equations (see [22]):

$$(2.7) \quad G_1 : f \rightarrow v, \text{ where } \Delta v = f \text{ in } \Omega, v = 0 \text{ on } \partial\Omega, \quad G_1 : L^2(\Omega) \rightarrow H^2(\Omega),$$

$$(2.8) \quad \begin{aligned} G_2 : g \rightarrow w, \text{ where } \Delta w = 0 \text{ in } \Omega, w = g \text{ on } \partial\Omega, \\ G_2 : H^s(\partial\Omega) \rightarrow H^{s+\frac{1}{2}}(\Omega) \quad (s \geq 1). \end{aligned}$$

It follows from Chapter 2 of [32] that the projection P_F onto the subspace F of (2.2), (2.3) has the following form:

$$(2.9) \quad P_F = \operatorname{grad} G_1 \operatorname{div}.$$

We denote by P_J and P_G the projections onto J and G .

If W is an open subset of \mathbb{R}^n , we denote by $S_{1,0}^m = S_{1,0}^m(W, \mathbb{R}^n)$ the set of all $p \in C^\infty(W, \mathbb{R}^n)$ such that for every compact set $O \subset W$ we have

$$(2.10) \quad |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{O,\alpha,\beta} (1 + |\xi|)^{m-|\alpha|}, \quad x \in O, \xi \in \mathbb{R}^n$$

for all $\alpha, \beta \in \mathbb{N}_+^n$. The elements of $S_{1,0}^m$ are called symbols (or full symbols) of order m . It is clear that $S_{1,0}^m$ is a Fréchet space with semi-norms given by the smallest constants which can be

used in (2.10) (i.e.,

$$\|p\|_{O,\alpha,\beta} = \sup_{x \in O} |(D_x^\beta D_\xi^\alpha p(x, \xi)) (1 + |\xi|)^{|\alpha|-m}|.$$

Let $p(x, \xi) \in S_{1,0}^m$. A pseudo-differential operator in an open set $W \subset \mathbb{R}^n$ is essentially defined by a Fourier integral operator (cf. [14], [31], [11]):

$$(2.11) \quad P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x, \xi) e^{ix \cdot \xi} \hat{u}(\xi) d\xi,$$

and denoted by OPS^m . Here $u \in C_0^\infty(W)$ and $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} u(y) dy$ is the Fourier transform of u . If there are smooth $p_{m-j}(x, \xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$, that is, $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r, |\xi| \geq 1$, and if

$$(2.12) \quad p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$$

in the sense that

$$(2.13) \quad p(x, \xi) - \sum_{j=0}^l p_{m-j}(x, \xi) \in S_{1,0}^{m-l-1},$$

for all l , then we say $p(x, \xi) \in S_{cl}^m$, or just $p(x, \xi) \in S^m$. We call $p_m(x, \xi)$ the principal symbols of $P(x, D)$.

An operator P is said to be an elliptic pseudodifferential operator of order m if for every compact $O \subset \Omega$ there exists a positive constant $c = c(O)$ such that

$$|p(x, \xi)| \geq c|\xi|^m, \quad x \in O, \quad |\xi| \geq 1$$

for any compact set $O \subset \Omega$. If P is a non-negative elliptic pseudodifferential operator of order m , then the spectrum of P lies in a right half-plane and has a finite lower bound $\tau(P) = \inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(P)\}$. We can modify $p_m(x, \xi)$ for small ξ such that $p_m(x, \xi)$ has a positive lower bound throughout and lies in $\{\lambda = re^{i\theta} \mid r > 0, |\theta| \leq \theta_0\}$, where $\theta_0 \in (0, \frac{\pi}{2})$. According to [11], the resolvent $(P - \lambda)^{-1}$ exists and is holomorphic in λ on a neighborhood of a set

$$W_{r_0, \epsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq r_0, \arg \lambda \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon], \operatorname{Re} \lambda \leq \tau(P) - \epsilon\}$$

(with $\epsilon > 0$). There exists a parametrix Q'_λ on a neighborhood of a possibly larger set (with $\delta > 0, \epsilon > 0$)

$$V_{\delta, \epsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq \delta \text{ or } \arg \lambda \in [\theta_0 + \epsilon, 2\pi - \theta_0 - \epsilon]\}$$

such that this parametrix coincides with $(P - \lambda)^{-1}$ on the intersection. Its symbol $q(x, \xi, \lambda)$ in local coordinates is holomorphic in λ there and has the form (cf. Section 3.3 of [11])

$$(2.14) \quad q(x, \xi, \lambda) \sim \sum_{l \geq 0} q_{-m-l}(x, \xi, \lambda)$$

where

$$(2.15) \quad \begin{aligned} q_{-m} &= (p_m(x, \xi) - \lambda)^{-1}, & q_{-m-1} &= b_{1,1}(x, \xi) q_{-m}^2, \\ \cdots, & q_{-m-l} &= \sum_{k=1}^{2l} b_{l,k}(x, \xi) q_{-m}^{k+1}, \cdots, \end{aligned}$$

with symbols $b_{l,k}$ independent of λ and homogeneous of degree $mk - l$ in ξ for $|\xi| \geq 1$. The semigroup e^{-tP} can be defined from P by the Cauchy integral formula (see p. 4 of [10]):

$$e^{-tP} = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (P - \lambda)^{-1} d\lambda,$$

where \mathcal{C} is a suitable curve in the positive direction around the spectrum of P .

It follows from [19] that the domain $\mathcal{D}(S)$ of the Stokes operator is dense in the Hilbert space J with the inner product of $[L^2(\Omega)]^n$, and the operator S is symmetric and positive definite in the Hilbert space J . The following proposition duo to Kozhevnikov (see, [19]):

Proposition 2.1. *The Stokes operator S is continuously invertible in the space J , and there exists a pseudodifferential operator K_{-1} on $\partial\Omega$ such that*

$$(2.16) \quad S^{-1}f = -(1/\mu)G_1[I - 2 \operatorname{grad}(I + G_2K_{-1}\gamma_0)\operatorname{div} G_1]f, \quad f \in J,$$

where the operators G_1, G_2 and γ_0 are defined in (2.7), (2.8) and (2.6), and I is an identity matrix.

3. LEMMAS

Now, we denote by A the extension of S^{-1} to all of $[L^2(\Omega)]^n$ that is given by the same formula as S^{-1} :

$$(3.1) \quad Af = -(1/\mu)G_1[I - 2 \operatorname{grad}(I + G_2K_{-1}\gamma_0)\operatorname{div} G_1]f, \quad f \in [L^2(\Omega)]^n.$$

Since $[L^2(\Omega)]^n = J \oplus F \oplus G$, any element $f \in [L^2(\Omega)]^n$ is thus uniquely determined by the column of its “coordinates” $(f_J, f_F, f_G)^T$, where T denotes the transpose. The operator A of (3.1) can be written correspondingly in the matrix form

$$(3.2) \quad Af = \begin{pmatrix} A_{JJ} & A_{JF} & A_{JG} \\ A_{FJ} & A_{FF} & A_{FG} \\ A_{GJ} & A_{GF} & A_{GG} \end{pmatrix} \begin{pmatrix} f_J \\ f_F \\ f_G \end{pmatrix},$$

where, for example, A_{JF} maps F into J .

Lemma 3.1. *The matrix (3.2) representing the operator A has the form*

$$(3.3) \quad A = \begin{pmatrix} S^{-1} & 0 & 0 \\ 0 & A_{FF} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. It follows from [19] that the matrix (3.2) representing the operator A has the form

$$A = \begin{pmatrix} S^{-1} & A_{JF} & 0 \\ 0 & A_{FF} & 0 \\ 0 & A_{GF} & 0 \end{pmatrix}.$$

According to the Proof of Lemma of [19], we know that $Af = 0$ for any $f \in G$. Thus we may assume $f_G \equiv 0$ in the above representation. Since A is a self-adjoint operator in $[L^2(\Omega)]^n$, we have

$$(3.4) \quad (Af, g) = (f, Ag) \quad \forall f, g \in [L^2(\Omega)]^n,$$

so that the above relation still holds for any $f, g \in J \oplus F$. Note that for any $f = f_J + f_F$, $g = g_J + g_F \in J \oplus F$,

$$(Af, g) = (S^{-1}f_J, g_J) + (A_{JF}f_F, g_J) + (A_{FF}f_F, g_F),$$

$$(f, Ag) = (f_J, S^{-1}g_J) + (f_F, A_{FJ}g_J) + (f_F, A_{FF}g_F).$$

Since S^{-1} , A_{FF} and A are self-adjoint operators on J , F and $J \oplus F$, respectively, we find by $A_{FJ} \equiv 0$ that

$$(A_{JF}f_F, g_J) = (f_F, A_{FJ}g_J) = 0, \quad \text{for all } f_F \in F \text{ and } g_J \in J,$$

which implies $A_{JF} = 0$. Similarly, we can show that $A_{GF} \equiv 0$. Hence, the matrix (3.2) representing the operator A is symmetric, and the desired result is proved. \square

By virtue of the compactness of G_1 and (3.1), we see that the operator A is compact on $[L^2(\Omega)]^n$. Since A is a self-adjoint nonnegative-definite operator with respect to the L^2 inner product, we get that A has an orthonormal basis of eigenfunctions $\{w_k\}_{k=1}^\infty$ corresponding to eigenvalues $\{\tau_k\}_{k=1}^\infty$ which satisfy

$$\tau_1 \geq \tau_2 \geq \cdots \geq \tau_k \geq \cdots \geq 0$$

and $\tau_k \rightarrow 0$ as $k \rightarrow +\infty$; in addition, the number 0 belongs to the spectrum of A . These properties allow one to define powers of A . We define A^2 by its action on $u \in \mathcal{D}(A)$:

$$A^2 u = \sum_{k=1}^{\infty} \tau_k^2 (u, w_k) w_k,$$

where (\cdot, \cdot) is the L^2 inner product.

It follows from (1) of §8 of [19] that A^2 can be represented as

$$(3.5) \quad A^2 f = \mu^{-2} G_1 (I + K_1) G_1 f, \quad f \in [L^2(\Omega)]^n,$$

where K_1 is an operator that is smoothing by one unit, i.e., K_1 maps $[L^2(\Omega)]^n$ into $[H^2(\Omega)]^n$. The operator A_{FF} has the representation (see, (2) of §8 of [19]):

$$(3.6) \quad A_{FF} f_F = A_{FF} \text{grad } p = \mu^{-1} \text{grad } G_1 (I + K_2) p,$$

where $f_F = \text{grad } p \in F$, $p \in H_0^1(\Omega)$; K_2 is a smoothing operator, i.e., an operator of order -1 ; $K_2 : H_0^1(\Omega) \rightarrow H^2(\Omega)$.

Let

$$(3.7) \quad \tilde{A}_{FF} p := \mu^{-1} G_1 (I + K_2) p, \quad p \in H_0^1(\Omega),$$

where K_2 is the same operator as in (3.6).

In [19], Kozhevnikov showed that the number $\lambda \neq 0$ and the vector $f = \text{grad } p \in F$ are an eigenvalue and corresponding eigenvector of the operator A_{FF} if and only if the pair (λ, p) constitutes an eigenvalue and eigenvector of the operator \tilde{A}_{FF} . From now on, we restrict the operators A^2 and A_{FF} on space $J \oplus F$ and still denote them by A^2 and A_{FF} .

Lemma 3.2. *The kernel spaces of the operators A^2 and \tilde{A}_{FF} are finite dimensional, and*

$$(3.8) \quad \dim(\ker A^2) = \dim(\ker \tilde{A}_{FF}).$$

Proof. Since the operators G_1^{+2} and G_1^{+1} are invertible, it follows from (3.5) and (3.7) that the kernels of A^2 and \tilde{A}_{FF} are finite-dimensional. If $A_{FF} f_F = 0$, where $f_F = \text{grad } p \in F$, $p \in H_0^1(\Omega)$, then, by (3.6) we have

$$\text{grad } G_1 (I + K_2) p = 0 \quad \text{in } \Omega,$$

so that

$$(3.9) \quad G_1 (I + K_2) p \equiv \text{const} \quad \text{in } \Omega.$$

According to the definition of G_1 (see (2.7)), we get $(G_1(I + K_2)p)|_{\partial\Omega} = 0$. It follows from (3.9) that $G_1(I + K_2)p \equiv 0$ in Ω , i.e., $\tilde{A}_{FF}p = 0$ in Ω . Conversely, if $p \in H_0^1(\Omega)$ and $\tilde{A}_{FF}p = 0$, i.e., $\mu^{-1}G_1(I + K_2)p = 0$ in Ω , then

$$A_{FF}f_F = \mu^{-1}\text{grad } G_1(I + K_2)p \equiv 0 \quad \text{in } \Omega.$$

Thus

$$(3.10) \quad \dim(\ker \tilde{A}_{FF}) = \dim(\ker A_{FF}).$$

We denote by M the above dimensional number.

Obviously, $\ker A_{FF} \subset \ker A_{FF}^2$. Now, let $x \in \ker A_{FF}^2$, i.e., $A_{FF}^2x = 0$. We claim that $A_{FF}x = 0$. Suppose by contradiction that $A_{FF}x \neq 0$. Let $\{\text{grad } p_j\}_{j=1}^\infty$ ($p_j \in H_0^1(\Omega)$) are orthonormal eigenvectors of A_{FF} corresponding to all non-zero eigenvalues $\{\alpha_j\}_{j=1}^\infty$, and let $\{r_j\}_{j=1}^M$ is an orthonormal basis of $\ker A_{FF}$. Let $x = \sum_{j=1}^\infty \beta_j(\text{grad } p_j) + \sum_{j=1}^M \tilde{\beta}_j r_j$. Then

$$A_{FF}x = \sum_{j=1}^\infty \alpha_j \beta_j (\text{grad } p_j) \neq 0,$$

so that $(\alpha_1 \beta_1, \dots, \alpha_j \beta_j, \dots) \neq (0, \dots, 0, \dots)$. Thus,

$$A_{FF}^2x = \sum_{j=1}^\infty \alpha_j^2 \beta_j (\text{grad } p_j) \neq 0$$

since $\{\text{grad } p_j\}$ is an orthonormal system. This contradicts the assumption $A_{FF}^2x = 0$. Therefore, the assertion $\ker A_{FF}^2 \subset \ker A_{FF}$ holds, and hence we have

$$(3.11) \quad \ker A_{FF}^2 = \ker A_{FF}.$$

Finally, since the operator S^{-1} is invertible in J , we have that $\ker A^2 = \ker A_{FF}^2$, so by (3.10) and (3.11), we obtain $\ker A^2 = \ker \tilde{A}_{FF}$. \square

4. ASYMPTOTIC EXPANSION

Proof of Theorem 1.1. Since $\ker A^2 = \ker \tilde{A}_{FF}$ which is a finite number M , we adjust the Jordan matrices A^2 and \tilde{A}_{FF} on the kernels of these operators by replacing the zero eigenvalue by a common constant, saying $\varrho > 0$. As pointed out in [19], this is equivalent to adding to A^2 and \tilde{A}_{FF} some finite-dimensional operators of order $-\infty$ since their kernels consist of infinitely smooth functions. We denote the finite-dimensional operators added by R and \tilde{R} .

By inverting the operators $A^2 + R$ and $\tilde{A}_{FF} + \tilde{R}$, we obtain that

$$(4.1) \quad (A^2 + R)^{-1} = \mu^2 G_1^{-2} + B_1,$$

$$(4.2) \quad (\tilde{A}_{FF} + \tilde{R})^{-1} = \mu G_1^{-1} + B_2,$$

where B_1 and B_2 are pseudodifferential operators of order 3 and 1, respectively.

The proof of (1.8) is broken up into a number of steps.

Step 1. We calculate the asymptotic expansion of the trace of semigroup $e^{-t(\mu G_1^{-1} + B_2)}$ as $t \rightarrow 0^+$. By (4.2) and the representation (3.7), we see that $\mu G_1^{-1} + B_2$ defined on $H_0^1(\Omega)$ is a pseudodifferential operator with symbol $\mu \sum_{j=1}^n \xi_j^2 + \sum_{m=0}^\infty p_{1-m}(x, \xi)$, where $p_{1-m}(x, \xi)$ is homogeneous in ξ of degree $1 - m$ for $|\xi| > 1$. It follows from Section 2 that the semigroup

$e^{-t(\mu G_1^{-1} + B_2)}$ can be defined from the resolvent operator $(\mu G_1^{-1} + B_2 - \lambda)^{-1}$ (defined on $H_0^1(\Omega)$) by the Cauchy integral formula

$$(4.3) \quad e^{-t(\mu G_1^{-1} + B_2)} = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (\mu G_1^{-1} + B_2 - \lambda)^{-1} d\lambda.$$

Note that the interior asymptotics are independent of the boundary condition; however, the boundary asymptotics depend on the Dirichlet boundary conditions (see, [3], [8] and [16]). The symbol $q(x, \xi, \lambda)$ of $(\mu G_1^{-1} + B_2 - \lambda)^{-1}$ in local coordinates is holomorphic in λ there and has the form

$$q(x, \xi, \lambda) \sim \sum_{l \geq 0} q_{-2-l}(x, \xi, \lambda), \quad \text{where } q_{-2} = (\mu \sum_{j=1}^n \xi_j^2 - \lambda)^{-1},$$

$$q_{-2-1} = b_{1,1}(x, \xi) q_{-2}^2, \dots, q_{-2-l} = \sum_{k=1}^{2l} b_{l,k}(x, \xi) q_{-2}^{k+1}, \dots;$$

with symbols $b_{l,k}$ independent of λ and homogeneous of degree $2k - l$ in ξ for $|\xi| \geq 1$. Thus

$$(4.4) \quad v_{-2}(t, x, \xi) = e^{-t\mu \sum_{j=1}^n \xi_j^2}, \quad v_{-2-l}(t, x, \xi) = \sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2}, \quad l \geq 1,$$

which implies

$$(4.5) \quad \begin{aligned} \text{Tr} \left(e^{-t(\mu G_1^{-1} + B_2)} \Big|_{\Omega'} \right) &= \text{Tr} \left[\left(\frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (\mu G_1^{-1} + B_2 - \lambda)^{-1} d\lambda \right) \Big|_{\Omega'} \right] \\ &= \int_{\Omega'} dx \int_{\mathbb{R}^n} \left[e^{-t\mu \sum_{j=1}^n \xi_j^2} + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) \right] d\xi \end{aligned}$$

for any $\Omega' \Subset \Omega$. Since $b_{1,k}(x, \xi) = -b_{1,k}(x, -\xi)$ for any $\xi \in \mathbb{R}^n$, we have

$$(4.6) \quad \int_{\mathbb{R}^n} \left(\sum_{k=1}^2 \frac{t^k}{k!} b_{1,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) d\xi = 0.$$

In the interior of Ω , for any $\Omega' \subset \Omega$ we find by a direct calculation that

$$(4.7) \quad \begin{aligned} \int_{\Omega'} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[e^{-t\mu \sum_{j=1}^n \xi_j^2} + \sum_{l=2}^{\infty} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) \right] d\xi \right\} dx \\ = \frac{1}{(4\pi\mu t)^{n/2}} \int_{\Omega'} dx + O(t) = \frac{|\Omega'|}{(4\pi\mu t)^{n/2}} + O(t) \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

This is a straightforward application of the Seeley calculus [28] (Note that

$$(4.8) \quad \int_{\Omega'} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{l=2}^{\infty} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) d\xi \right\} dx = O(t) \quad \text{as } t \rightarrow 0^+.$$

More importantly, (4.7) is still valid for any n -dimensional normal coordinate patch W covering a patch of $W \cap \partial\Omega$.

It remains to consider the boundary asymptotics. Let $x = (x'; x_n)$ be local coordinates for Ω near $\partial\Omega$. If \mathfrak{E} is a local frame on $\partial\Omega$; extend \mathfrak{E} to a local frame in a neighborhood of $\partial\Omega$

by parallel transport along the geodesic normal rays (see, p.1101 of [21]). For any small n -dimensional normal coordinate patch W covering a patch of $W \cap \partial\Omega$, we integrate for $x_n \in [0, \epsilon]$ to define the boundary contribution (see [3], [8], [9], [16] or Remark 4.1):

$$(4.9) \quad - \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n \xi_n} \left\{ \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \left[e^{-t\mu \sum_{j=1}^n \xi_j^2} + \sum_{l=1}^\infty \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) \right] d\xi' \right\} d\xi_n,$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$. It is completely similar to the previous argument that

$$(4.10) \quad \frac{1}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n \xi_n} \left\{ \int_{\mathbb{R}^{n-1}} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) d\xi' \right\} d\xi_n = 0$$

and

$$(4.11) \quad \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n \xi_n} \left\{ \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \times \left[\sum_{l=2}^\infty \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) \right] d\xi' \right\} d\xi_n = O(t) \quad \text{as } t \rightarrow 0^+.$$

Note that

$$(4.12) \quad \begin{aligned} & \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n \xi_n} \left[\int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} e^{-t\mu \sum_{j=1}^n \xi_j^2} d\xi' \right] d\xi_n \\ &= \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{1}{(4\pi\mu t)^{\frac{n-1}{2}}} dx' \cdot \frac{1}{(2\pi)} \int_{-\infty}^\infty e^{2ix_n \xi_n} e^{-t\mu \xi_n^2} d\xi_n \\ &= \int_0^\epsilon \frac{1}{(4\pi\mu t)^{\frac{n}{2}}} e^{-\frac{x_n^2}{4\mu t}} dx_n \int_{W \cap \partial\Omega} dx' \\ &= \frac{1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi\mu t)^{\frac{n-1}{2}}} + O(t) \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

We find by (4.10)–(4.12) that

$$(4.13) \quad \begin{aligned} & - \int_0^\epsilon dx_n \int_{W \cap \partial\Omega} \frac{dx'}{(2\pi)^n} \int_{-\infty}^\infty e^{2ix_n \xi_n} \left\{ \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \left[e^{-t\mu \sum_{j=1}^n \xi_j^2} + \sum_{l=1}^\infty \left(\sum_{k=1}^{2l} \frac{t^k}{k!} b_{l,k}(x, \xi) e^{-t\mu \sum_{j=1}^n \xi_j^2} \right) \right] d\xi' \right\} d\xi_n \\ &= -\frac{1}{4} \cdot \frac{|W \cap \partial\Omega|}{(4\pi\mu t)^{\frac{n-1}{2}}} + O(t) \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Combining (4.7) and (4.13) we get

$$(4.14) \quad \text{Tr } e^{-t(\mu G_1^{-1} + B_2)} = (4\pi\mu t)^{-n/2} \left[|\Omega| - \frac{1}{4} \sqrt{4\pi\mu t} |\partial\Omega| + O(t) \right] \quad \text{as } t \rightarrow 0^+.$$

Step 2. From (4.1) we know that the full symbol of the pseudodifferential operator $\mu^2 G_1^{-2} + B_1$ (defined on $[H_0^1(\Omega)]^n$) is $\mu^2 \left(\sum_{j=1}^n \xi_j^2 \right)^2 I + \sum_{m=0}^\infty p_{3-m}(x, \xi)$, where the matrix $p_{3-m}(x, \xi)$ is

homogeneous in ξ of degree $3 - m$ when $|\xi| > 1$ for its each entry. A direct calculation shows

$$(4.15) \quad \begin{aligned} \text{Tr}(e^{-t(\mu^2 G_1^{-2} + B_1)}) &= \text{Tr} \left(\frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (\mu^2 G_1^{-2} + B_1 - \lambda I)^{-1} d\lambda \right) \\ &= \int_{\Omega} \left\{ \text{Tr} \left[\int_{\mathbb{R}^n} e^{-t\mu^2 (\sum_{j=1}^n \xi_j^2)^2} I + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} c_{l,k}(x, \xi) e^{-t\mu^2 (\sum_{k=1}^n \xi_k^2)^2} \right) \right] d\xi \right\} dx, \end{aligned}$$

where the symbol-matrix $c_{l,k}(x, \xi)$ is independent of λ and its entry is homogeneous of degree $4k - l$. Note that

$$(4.16) \quad \int_{\mathbb{R}^n} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} c_{1,k}(x, \xi) e^{-t\mu^2 (\sum_{k=1}^n \xi_k^2)^2} \right) d\xi = 0$$

since $c_{1,k}(x, \xi)$ is odd with respect to ξ . Similarly to the argument in step 1, we have

$$\begin{aligned} \text{Tr}(e^{-t(\mu^2 G_1^{-2} + B_1)}) &= \int_{\Omega} \left\{ \text{Tr} \left[\int_{\mathbb{R}^n} e^{-t\mu^2 (\sum_{j=1}^n \xi_j^2)^2} I + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{2l} \frac{t^k}{k!} c_{l,k}(x, \xi) e^{-t\mu^2 (\sum_{k=1}^n \xi_k^2)^2} \right) \right] d\xi \right\} dx \\ &= \frac{n}{(4\pi\mu t)^{n/4}} \left[\frac{|\Omega|}{4^{\frac{n}{2}} \pi^{\frac{n-2}{4}} \Gamma(\frac{n+2}{4})} - \frac{|\partial\Omega|(\mu t)^{\frac{1}{4}}}{\sqrt{2} \cdot 4^{\frac{n}{2}} \pi^{\frac{n-1}{4}}} + O(t^{\frac{1}{2}}) \right] \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Here we have used the trace of the matrix, by which the coefficients of asymptotic expansion are multiplied by n .

It is well-known (see, for example, (5.22) of p 247 of [30], [10]) that for $\lambda \geq 0$,

$$e^{-t\sqrt{\lambda}} = \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{-t^2/(4s)} e^{-s\lambda} ds,$$

i.e., the Laplace transform of $\frac{t}{\sqrt{4\pi s^3}} e^{-t^2/(4s)}$ is $e^{-t\sqrt{\lambda}}$. By applying the spectral theorem, we get that for all $t > 0$,

$$(4.17) \quad e^{-t\sqrt{\mu^2 G_1^{-2} + B_1}} = \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{-t^2/(4s)} e^{-s(\mu^2 G_1^{-2} + B_1)} ds.$$

Therefore, we have

$$\begin{aligned} (4.18) \quad \text{Tr}(e^{-t(A^2 + R)^{-1/2}}) &= \text{Tr}(e^{-t\sqrt{\mu^2 G_1^{-2} + B_1}}) \\ &= \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{-t^2/(4s)} \left[\text{Tr}(e^{-s(\mu^2 G_1^{-2} + B_1)}) \right] ds \\ &= \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{-t^2/(4s)} \frac{n}{(4\pi\mu s)^{n/4}} \left[\frac{|\Omega|}{4^{\frac{n}{2}} \pi^{\frac{n-2}{4}} \Gamma(\frac{n+2}{4})} - \frac{|\partial\Omega|(\mu s)^{\frac{1}{4}}}{\sqrt{2} \cdot 4^{\frac{n}{2}} \pi^{\frac{n-1}{4}}} + O(s^{\frac{1}{2}}) \right] ds \\ &= \frac{n}{(4\pi\mu t)^{n/2}} \left[|\Omega| - \frac{1}{4} \sqrt{4\pi\mu t} |\partial\Omega| + O(t) \right] \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Step 3. Recall that the kernel spaces of A^2 and \tilde{A}_{FF} have the same dimensional number M . As in proof of Lemma 3.2, let $\{\phi_j\}_{j=1}^{\infty}$ (respectively, $\{p_j\}_{j=1}^{\infty}$) are the orthonormal eigenvectors corresponding to all non-zero eigenvalues $\{\alpha_j\}_{j=1}^{\infty}$ (respectively, $\{\beta_j\}_{j=1}^{\infty}$) of A (respectively, \tilde{A}_{FF}), and $\{\tilde{\phi}_j\}_{j=1}^M$ (respectively, $\{\tilde{p}_j\}_{j=1}^M$) is an orthonormal basis of A (respectively, of \tilde{A}_{FF}).

Then the operators $A^2 + R$ and $\tilde{A}_{FF} + \tilde{R}$ have the following representations:

$$\begin{aligned} A^2 + R &= \sum_{j=1}^{\infty} \alpha_j^2 \phi_j + \sum_{j=1}^M \varrho^2 \tilde{\phi}_j, \\ \tilde{A}_{FF} + \tilde{R} &= \sum_{j=1}^{\infty} \beta_j p_j + \sum_{j=1}^M \varrho \tilde{p}_j. \end{aligned}$$

Furthermore, the semigroups can be represented as

$$(4.19) \quad e^{-t(A^2+R)^{-\frac{1}{2}}} = \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} \phi_j + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \tilde{\phi}_j,$$

$$(4.20) \quad e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}} = \sum_{j=1}^{\infty} e^{-t\frac{1}{\beta_j}} p_j + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \tilde{p}_j;$$

in addition, the fundamental solutions of $e^{-t(A^2+R)^{-\frac{1}{2}}}$ and $e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}}$, respectively, are

$$K_1(t, x, y) = \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} \phi_j(x) \otimes \phi_j(y) + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \tilde{\phi}_j(x) \otimes \tilde{\phi}_j(y)$$

and

$$K_2(t, x, y) = \sum_{j=1}^{\infty} e^{-t\frac{1}{\beta_j}} p_j(x) \otimes p_j(y) + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \tilde{p}_j(x) \otimes \tilde{p}_j(y)$$

with uniform convergence on compact figures of $(0, \infty) \times \Omega \times \Omega$, and the traces $\text{Tr}(e^{-t(A^2+R)^{-\frac{1}{2}}})$ and $\text{Tr}(e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}})$ are easily evaluated as (see, for example, [26], [4])

$$\begin{aligned} \text{Tr}(e^{-t(A^2+R)^{-\frac{1}{2}}}) &= \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} \int_{\Omega} |\phi_j(x)|^2 dx + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \int_{\Omega} |\tilde{\phi}_j(x)|^2 dx \\ &= \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} + \sum_{j=1}^M e^{-t\frac{1}{\varrho}}, \\ \text{Tr}(e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}}) &= \sum_{k=1}^{\infty} e^{-t\frac{1}{\beta_j}} \int_{\Omega} |p_j(x)|^2 dx + \sum_{l=1}^M e^{-t\frac{1}{\varrho}} \int_{\Omega} |\tilde{p}_j(x)|^2 dx \\ &= \sum_{j=1}^{\infty} e^{-t\frac{1}{\beta_j}} + \sum_{j=1}^M e^{-t\frac{1}{\varrho}}. \end{aligned}$$

Note that $\{\alpha_j\}_{j=1}^{\infty}$ (respectively, $\{\beta_j\}_{j=1}^{\infty}$) just are all non-zero eigenvalues of A (respectively, of \tilde{A}_{FF} (i.e., of A_{FF})). According to (3.3), we get

$$\begin{aligned} (4.21) \quad \text{Tr}(e^{-tS}) &= \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} - \sum_{j=1}^{\infty} e^{-t\frac{1}{\beta_j}} \\ &= \sum_{j=1}^{\infty} e^{-t\frac{1}{\alpha_j}} + \sum_{j=1}^M e^{-t\frac{1}{\varrho}} - \sum_{j=1}^{\infty} e^{-t\frac{1}{\beta_j}} - \sum_{j=1}^M e^{-t\frac{1}{\varrho}} \\ &= \text{Tr}(e^{-t(A^2+R)^{-1/2}}) - \text{Tr}(e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}}) \end{aligned}$$

In view of

$$K(t, x, y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} u_j(x) \otimes u_j(y),$$

we have

$$\mathrm{Tr}(e^{-tS}) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \int_{\Omega} |u_j(x)|^2 dx = \sum_{j=1}^{\infty} e^{-t\lambda_j},$$

where $\{u_j\}_{j=1}^{\infty}$ are the orthonormal eigenvectors corresponding to the Stokes eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$. By (4.21), (4.14) and (4.18) we obtain

$$\begin{aligned} (4.22) \quad \sum_{j=1}^{\infty} e^{-t\lambda_j} &= \mathrm{Tr}(e^{-t(A^2+R)^{-1/2}}) - \mathrm{Tr}(e^{-t(\tilde{A}_{FF}+\tilde{R})^{-1}}) \\ &= \mathrm{Tr}(e^{-t(\mu G_1^{-2}+R)^{-1/2}}) - \mathrm{Tr}(e^{-t(\mu G_1^{-1}+\tilde{R})^{-1}}) \\ &= \frac{n-1}{(4\pi\mu t)^{n/2}} \left[|\Omega| - \frac{1}{2} \sqrt{4\pi\mu t} |\partial\Omega| + O(t) \right] \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

□

Remark 4.1. (4.14) can also be obtained by McKean-Singer's method (see, §5 of [24]): Let $\mathcal{M} = \Omega \cup (\partial\Omega) \cup \Omega^*$ be the (closed) double of Ω , and Q the double to \mathcal{M} of the pseudodifferential operator $\mu G_1^{-1} + B_2$. Define Q^- to be $Q|_{C^\infty(\bar{\Omega})}$ subject to $u = 0$ on $\partial\Omega$. The symbol of $\mu G_1^{-1} + B_2$ on \mathcal{M} jumps as x crosses $\partial\Omega$, but $\frac{\partial u}{\partial t} = Qu$ still has a nice fundamental solution \mathfrak{F} of class $C^\infty([0, \infty) \times (\mathcal{M} - \partial\Omega)^2) \cap C^1(\mathcal{M}^2)$, approximable even on $\partial\Omega$, and the fundamental solution \mathfrak{F}^- of $\frac{\partial u}{\partial t} = Q^-u$ can be expressed on $(0, \infty) \times \Omega \times \Omega$ as

$$(4.23) \quad \mathfrak{F}^-(t, x, y) = \mathfrak{F}(t, x, y) - \mathfrak{F}(t, x, y^*),$$

y^* being the double of $y \in \Omega$ (see, p. 53 of [24]). We pick a self-double patch W of \mathcal{M} covering a patch $W \cap \partial\Omega$ endowed (see the diagram on p. 53 of [24]) with local coordinates x such that $\epsilon > x_n > 0$ in $W \cap \Omega$; $x_n = 0$ on $W \cap \partial\Omega$; $x_n(x^*) = -x_n(x)$; and the positive x_n -direction is perpendicular to $\partial\Omega$. This produces the following effect that

$$\begin{aligned} \delta_{jk}(x^*) &= -\delta_{jk}(x) \quad \text{for } j < k = n \text{ or } k < j = n \\ &= \delta_{jk}(x) \quad \text{for } j, k < n \text{ or } j = k = n, \\ \delta_{jk}(x) &= 0 \quad \text{for } j < k = n \text{ or } k < j = n \text{ on } \partial\Omega. \end{aligned}$$

For any $\Omega' \subset \Omega$, by the technique of step 1 we see that (4.7) still holds. For any n -dimensional normal coordinate patch W covering a patch $W \cap \partial\Omega$, it is completely similar to (4b) (5a) (5b) (5c) of p 55 of [24] that

$$\begin{aligned} \int_{W \cap \Omega} \Phi(t, x, x) dx &= \frac{1}{(4\pi\mu t)^{\frac{n}{2}}} \left[\int_{W \cap \Omega} 1 dx + O(t) \right], \\ \int_{W \cap \Omega} \Phi(t, x, x^*) dx &= \frac{1}{4} \cdot \frac{1}{(4\pi\mu t)^{\frac{n-1}{2}}} \left[\int_{W \cap \partial\Omega} 1 dx + O(t) \right], \\ \int_{W \cap \Omega} \left[\mathfrak{F}^-(t, x, x) - \Phi(t, x, x) \right] dx &= O\left(\frac{1}{(4\pi\mu t)^{\frac{n}{2}-1}} \right), \\ \int_{W \cap \Omega} \left[\mathfrak{F}^-(t, x, x^*) - \Phi(t, x, x^*) \right] dx &= O\left(\frac{1}{(4\pi\mu t)^{\frac{n}{2}-1}} \right), \end{aligned}$$

where $\Phi(t, x, y) = \frac{1}{(4\pi\mu t)^{\frac{n}{2}}} e^{-\frac{|y-x|^2}{4\mu t}}$ (it is just the first term of the Levi's sum in the expansion of $\mathfrak{F}^-(t, x, y)$, see §3 of [24]). Combining these, (4.23) and (4.7), we again get (4.14).

Now, we use the heat invariants which have been obtained from Theorem 1.1 to finish the proof of Corollary 1.2.

Proof of Corollary 1.2. By Theorem 1.1, we know that the first two coefficients $|\Omega|$ and $|\partial\Omega|$ of the asymptotic expansion in (1.8) are Stokes invariants, i.e., $|\Omega| = |B_r|$ and $|\partial\Omega| = |\partial B_r|$. Thus $\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} = \frac{|\partial B_r|}{|B_r|^{(n-1)/n}}$. According to the classical isometric inequality (which states that for any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, the following inequality holds:

$$\frac{|\partial\Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}.$$

Moreover, equality obtains if and only if Ω is a ball), we immediately get $\Omega = B_r$. \square

Remark 4.2. By applying the Tauberian theorem (see, for example, Theorem 15.3 of p. 30 of [18]) for the first term on the right side of (1.8), we immediately get the Weyl-type law (1.6) for the Stokes eigenvalues.

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REFERENCES

1. K. Abe and Y. Giga, *Analyticity of the Stokes semigroup in spaces of bounded functions*, Acta Math., 211(2013), 1-46.
2. K. I. Babenko, *On the asymptotic behavior of the eigenvalues of linearized Navier-Stokes equations*, Dokl. Akad. Nauk SSSR 263 (1982), 521-525, English transl. Soviet Math. Dokl., 25 (1982), 359-364.
3. T. P. Branson and P. B. Gilkey, *The asymptotics of the Laplacian on a manifold with boundary*, Comm. Partial Differential Equations, 15(1990), 245-272.
4. B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Science Press, Beijing, American Mathematical Society, Providence, RI, 2006.
5. A. J. Chorin and J. E. Marsden, *A mathematical introduction to fluid mechanics*, Springer-Verlag, New York, 1990.
6. P. Constantin and C. Foias, *Navier-Stokes Equations*. The University of Chicago Press, 1988.
7. S. A. Fulling, editor, *Heat Kernel Techniques and Quantum Gravity*, Winnipeg, Canada 1994, published as vol. 4 of Discourses in Mathematics and Its Applications, Dept. of Math., Texas A & M University, College Station, TX, 1995.
8. P. Gilkey, *Recursion relations and the asymptotic behavior of the eigenvalues of the Laplacian*, Compositio Math., 38(1979), 201-240.
9. P. B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Differential Geometry, 10(1975), 601-618.
10. H. Gimperlein and G. Grubb, *Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators*, J. Evolution Eq., 14 (2014), 49-83.

11. G. Grubb, *Functional calculus of pseudo-differential boundary problems*, Birkhäuser, Boston, 1986.
12. G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics, vol.252, Springer, New York, 2009.
13. L. Hörmander, *The analysis of partial differential operators III*, Springer-Verlag, Berlin Heidelberg New York, 1985.
14. L. Hörmander, *The analysis of partial differential operators IV*, Springer-Verlag, Berlin Heidelberg New York, 1985.
15. M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly (Slaught Mem. Papers, no. 11), 73(4)(1966), 1-23.
16. G. Kennedy, R. Critchley, and J. S. Dowker, *Finite temperature field theory with boundaries: Stress tensor and surface action renormalization*, Annals of Physics 125(1980), 346-400.
17. N. D. Kopachevsky, S. G. Krein, *Operator approach to linear problems of hydrodynamics*. Vol. 1, Oper. Theory Adv. Appl., vol. 128, Birkhäuser Verlag, Basel, 2001.
18. J. Korevaar, *Tauberian Theory: A Century of Developments*, Springer-Verlag, Berlin, Heidelberg, 2004.
19. A. N. Kozhevnikov, *On the operator of the linearized steady-state Navier-Stokes problem*, Math. USSR Sbornik, no.1, 53(1986), 1-16.
20. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Nauka, Moscow 1970; English transl. Gordon and Breach, New York 1969.
21. J. M. Lee and G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math., 42(1989), 1097-1112.
22. J. L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer-Verlag, Berlin-Heidelberg, 1972.
23. G. G. Lorentz, *Beweis des Gausschen Integralsatzes*, Math. Z., 51(1949), 61-81.
24. H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry, 1(1967), 43-69.
25. G. Metivier, *Valeurs propres des operateurs definis sur la restriction de systems variationnels a des sous-espaces*, J. Math. Pures Appl. 57 (1978), 133-156.
26. S. Minakshisundaram, *Eigenfunctions on Riemannian manifolds*, J. Indian Math. Soc., 17(1953), 158-165.
27. L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev. 9(3) (1967), 453-488.
28. R. Seeley, *Complex powers of an elliptic operator*, Proceedings of Symposia in Pure Mathematics 10(1967), 288-307.
29. V. A. Solonnikov and V. E. Shchadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov., 125(1973), 196-210; English transl. in Proc. Steklov Inst. Math., 125(1973).
30. M. E. Taylor, *Partial differential equations I*, Springer-Verlag, 1996.
31. M. E. Taylor, *Partial differential equations II*, Appl. Math. Sci., vol. 116, Springer-Verlag, New York, 1996.
32. R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, Amsterdam, North-Holland, 1984.
33. H. Weyl, *Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung*, J. Reine Angew. Math., 141(1912), 1-11.